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An Analysis of Grasp Quality Measures for the Application of Sheet Metal Parts Grasping

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Abstract Acquiring a qualitative force-closure grasp requires the determination of feasible contact points on the object based on a defined criterion. The determination must be fast in order to implement feasible synthesis algorithms. Moreover, grasping of sheet metal parts has further requirements derived from its geometry and clamping tools. This paper presents a grasp quality analysis for the application of sheet metal parts. Moreover, a novel grasp quality measure approach is proposed based on standard deviation computation of the contact's coordinates. The proposed measure is frame invariant, simple to implement, and has low computational complexity. A comparative analysis over other measures is presented. Further, a stress analysis was performed to show that the proposed criterion yields low stress on the sheet metal part compared to other criteria. Simulations show advantage to grasp synthesis with the proposed quality measure.

Keywords Grasp quality measure · Grasp synthesis · Sheet metal parts

1 Introduction

Grasp synthesis involves the process of determining the position of the contact points on the grasped object, the forces to be applied and the configuration of the end-effector in order to perform the grasp. The problem of determining the best contact points for the fingers to grasp a given object has been widely approached by different methods. The common method in off-line planning is acquiring a mesh representation of the object using an existing CAD model [26, 27], and searching for a feasible grasp with a known number of fingers [18, 31]. This method yields many possible grasp combinations. Hence, a grasp quality measure is defined and the best grasp chosen is the one that maximizes the quality measure.

There are many grasp quality measures used in various grasp synthesis algorithms. Ferrari and Canny [6] and Li and Sastry [11] introduced a quality measure based on the external wrench to be resisted; the former introduced a general measure based on the largest wrench magnitude that the grasp can resist, and the latter used a task-oriented quality measure defined by the specific wrenches applied to the object during execution of the task. The measure introduced by Ferrari and Canny is the most common one, measuring the radius of the largest ball that could be fully contained in the convex-hull of the grasp wrenches (this method will be further presented in detail). However, this method has a computational complexity of $O(h \log h)$ [25], where h is the number of wrenches generated by the contacts (discretizing the friction cones at the contact points, as will be discussed in Section 2, increases h significantly). This complexity results in high computational runtime, and makes it hard to acquire efficient grasp synthesis. Lin et al. [12] presented a frame-invariant quality measure for compliant grasps and fixtures, and presented an application for the planning of grasps and fixtures. Mirtich and Canny [15] proposed a computational method for two- and three-finger optimal grasps using an optimal-

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ity criterion based on decoupled wrenches, i.e., taking into account pure forces or pure moments. Li and Sastry also introduced a quality that measures how far the grasp configuration is from reaching singularity.

The quality measures mentioned above are based on the position of the fingers and the force direction they apply; other quality measures are based on geometric criteria where the distribution of the fingers on the objects is maximized. A common quality measure criterion for 3-finger planar and spatial objects is based on the area of the triangle formed by the three contact points [5,15]; it assesses the distribution of the contact points. For grasping planar objects with more than 3 fingers, the area of the convex hull of the contact points is computed. In the case of grasping spatial objects with more than 3 fingers the computation of this grasp quality gets more complex. This problem involves high computational complexity and occasionally multiple possible solutions. This measure will be discussed broadly in Section 3. Kim et al. [9] presented the stability grasp index, which defines a polygon created of the contact points and measures the deviation of the polygons angles from a regular polygon; this implies the distribution of the polygon on the object.

In off-line grasp synthesis algorithms [29], all or arbitrary grasps are sampled over a given object and checked for force closure and a quality measure threshold. Such search suffers from high computation time, mainly when increasing the desired contact points. For example, in whole arm grasping [30], the wrapping of the arm or fingers on the object is discretized to numerous contact points [34]. In such case, fast computation of a quality measure for each sampled grasp is crucial.

This paper presents a novel quality measure based on the standard deviation of the contact points. The measure termed *Standard Deviation based Quality Measure* (SDQM) is a coordinate frame invariant criterion that evaluates the distribution of the contact points on the grasped object. The proposed criterion is shown to be simple to implement with extremely low computational runtime compared to other criteria. Moreover, we show that the method is insensitive to the increase of fingers in terms of performance (i.e., runtime) unlike other methods.

We introduce an application of the SDQM to grasping of Sheet Metal Parts (SMPs). SMPs are curved thin and flat objects formed by bending, cutting, and stamping sheet metal plates. The grasp and fixture of an SMP is usually done with clamps or suction cups. Moreover, its geometry requires well distributed fixture points on the surface of the SMP to prevent distortion and damage. There is much work in the field of SMP fixturing [1,3,4,7,21]. We propose the use of the SDQM criterion for grasp synthesis of SMPs as a simple to implement method with low computational cost over other criteria. The SDQM provides a measure for the distribution of the clamps on the sheet, which is essential for SMP grasping. Further we show a comparative stress analysis to examine the von Mises stress developed with reference to the grasp criteria. To the authors' knowledge, no previous stress analysis was performed with regard to grasp quality measures. It should be noted that we deal with the grasping of known objects, that is, a detailed polygonal mesh of the object exists.

This paper is organized as follows. Section 2 presents some grasping background to be used by the quality measures to be presented. In Section 3 we discuss several common quality measures used in this paper for comparison and analysis. Section 4 discusses in detail the novel SDQM criterion and in Section 4.2 planar test cases are presented. In Section 5 we apply the SDQM to a special case of grasping SMPs and present analysis and performance results. Conclusions are given in Section 6.

2 Preliminaries

In this section we review some relevant grasping fundamentals. We present the grasp model we use and discuss the notions of force-closure.

2.1 Grasping Model

Forces and torques can be represented as wrench vectors in the wrench space. A wrench is a d-dimensional vector where $d = 3$ for a 2D grasp and $d = 6$ in the 3D case, respectively. The wrench is denoted as $\mathbf{w} = \left(\mathbf{f}^T\ \mathbf{u}^T\right)^T \in \mathbb{R}^d$ where $\mathbf{f}$ is the force vector and $\mathbf{u}$ is the torque vector. Furthermore, a wrench applied at the contact point, $\mathbf{p_i}$, can be described as $\mathbf{w_i} = \left(\mathbf{f}^T\ (\mathbf{p}_i \times \mathbf{f_i})^T\right)^T$, where $\mathbf{p_i}$ is represented in the object coordinate frame. Friction exists at the contacts between the fingertips of the end-effector and the object's surface. Friction can be represented by the simple Coulomb friction model. In this model, forces exerted at the contact point must lie within a cone centered about the surface normal. This notion is known as the *Friction Cone* (FC) and is given as

$$FC = \left\{ \begin{pmatrix} \mathbf{f_{i,1}} \\ \mathbf{f_{i,2}} \\ \mathbf{f_{i,3}} \end{pmatrix} : \left| \sqrt{\mathbf{f_{i,2}}^2 + \mathbf{f_{i,3}}^2} \right| \leq \mu \mathbf{f_{i,1}},\ \forall \mathbf{f_{i,1}} > 0 \right\}, \quad (1)$$

where $\mathbf{f_{i,1}}$ is the normal component, $\mathbf{f_{i,2}}$ and $\mathbf{f_{i,3}}$ are the tangential components at the contact point, and μ

is the coefficient of friction. The FC is non-linear and therefore can be approximated with an s-sided convex polytope and every force $\mathbf{f_i}$ exerted within the FC can be represented by a linear combination of the unit vectors $\hat{\mathbf{f}}_{\mathbf{ik}} \in \partial FC$ (primitive forces) constructing the linearized friction cone (LFC),

$$LFC = \left\{ \mathbf{f_i} : \mathbf{f_i} = \sum_{k=1}^{s} a_{ik} \hat{\mathbf{f}}_{\mathbf{ik}}, \; a_{ik} \geq 0 \right\} \tag{2}$$

where $LFC \subset FC$ and a_{ik} are nonnegative coefficients [13]. The $\hat{}$ sign denotes a unit vector. The associated wrenches can be expressed by the primitive forces as

$$\mathbf{w_i} = \sum_{k=1}^{s} a_{ik} \mathbf{w_{ik}} = \sum_{k=1}^{s} a_{ik} \begin{pmatrix} \hat{\mathbf{f}}_{\mathbf{ik}} \\ \mathbf{p_i} \times \hat{\mathbf{f}}_{\mathbf{ik}} \end{pmatrix} \tag{3}$$

where $\mathbf{w_{ik}}$ are the primitive wrenches associated with the primitive forces.

The geometry of an n-finger grasp can be represented by the locations $\mathcal{P} = \{\mathbf{p_1}, \ldots, \mathbf{p_n}\}$ of all contact points and the normals $\mathcal{N} = \{\mathbf{n_1}, ..., \mathbf{n_n}\}$. Equivalently, we can represent the grasp using the matching wrenches applied at the contact points represented in the object coordinate frame $\tilde{\mathcal{W}} = \{\mathbf{w_1}, ..., \mathbf{w_n}\}$. If we consider the friction cones, the wrench set can be expressed by the primitive wrenches

$$\mathcal{W} = \{\mathbf{w_{11}}, \mathbf{w_{12}}, ...\mathbf{w_{1s}}, ..., \mathbf{w_{n1}}, \mathbf{w_{n2}}, ..., \mathbf{w_{ns}}\} . \tag{4}$$

The effect of the contact forces $\mathbf{f_1}, ..., \mathbf{f_n}$ at contact points $\mathbf{p_1}, ..., \mathbf{p_n}$ on the object are given by the grasp map G such that the net wrench $\mathbf{w_0}$ is given by

$$\mathbf{w_0} = G \begin{bmatrix} \mathbf{f_1} \\ \vdots \\ \mathbf{f_n} \end{bmatrix} . \tag{5}$$

In the case of point contact with friction, the grasp map G is given by [17]

$$G = \begin{bmatrix} G_1 \cdots G_n \end{bmatrix} \in \mathbb{R}^{6 \times n} \tag{6}$$

where

$$G_i = \begin{bmatrix} R_i & 0 \\ \check{\mathbf{p}}_{\mathbf{i}} R_i & R_i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{7}$$

$R_i \in SO(3)$ is a rotation matrix from contact point i's reference frame to the objects reference frame, and $\check{\mathbf{p}}_{\mathbf{i}}$ is the skew-symmetric matrix representation of $\mathbf{p_i}$ given by

$$\check{\mathbf{p}}_{\mathbf{i}} = \begin{bmatrix} 0 & -\mathbf{p_{i_z}} & \mathbf{p_{i_y}} \\ \mathbf{p_{i_z}} & 0 & -\mathbf{p_{i_x}} \\ -\mathbf{p_{i_y}} & \mathbf{p_{i_x}} & 0 \end{bmatrix} . \tag{8}$$

Based on the model of the grasp, in the next subsection we present the notion of force closure.

2.2 Force Closure

A grasp is said to be force-closure if it is possible to apply wrenches at the contacts such that any external forces and torques applied to the object can be counter-balanced by the contact forces. A system of wrenches can achieve force-closure when they positively span the entire wrench space. Hence, any external load can be balanced by a non-negative combination of the wrenches. The Convex-Hull (CH) is mostly used to analyze the grasp and to determine whether it is force-closure[24]. The convex hull of the system of contact wrenches is denoted as the *Grasp Wrench Set* (GWS). After the GWS is defined, we use it to check if the CH positively spans the entire wrench space, meaning whether the grasp is force-closure. We check positive span to ensure positive grip (non-sticky fingers).

Theorem 1 *[16, 17, 24] A necessary and sufficient condition for a set of wrenches $\mathcal{W}$ to achieve force-closure is that the origin of $\mathbb{R}^d$ lies in the interior of the convex hull of the contact primitive wrenches. Meaning,*

$$O \in \; interior(CH(\mathcal{W})) \; . \tag{9}$$

The above Theorem defines whether a grasp is force closure. A practical method for implementing this is presented in [27]. Based on the notions presented in this section, in the next section we review several known grasp quality measures.

3 Grasp Quality Measures

It is common to classify grasp quality measures into two categories [5]; those that take solely the geometry of the grasp disregarding the end-effectors configurations and the ones that evaluate the end-effectors ability to perform the grasp. In this paper we will focus only on the former category. In particular, we will observe and compare our proposed quality measure to the following common measures [28]:

1. **Area of grasp polygon (AGP)**. The AGP quality measure was first related to 3-finger grasps. In [15] it was shown that the larger the triangle formed by the 3 contact points on the object is, the more robust the grasp is. That is, with the same finger forces the grasp can resist larger external torques. Formally, the AGP measure for planar and spatial 3-fingers grasp is calculated as

$$Q_{AGP} = Area(Triangle(\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3})) \ . \tag{10}$$

For planar grasps with more than 3 fingers, the grasp is evaluated by the area of the polygon formed by the n contact points. However, for non-convex polygons, there are several area solutions and therefore we usually calculate the area of the convex-hull (CH) formed by the contact points. That is,

$$Q_{AGP} = Area(CH(\mathbf{p_1}, ..., \mathbf{p_n})) \ . \tag{11}$$

For spatial grasps with more than 3 fingers the problem becomes more complex. Supuk et al. [33] proposed a method in which 3 fingers are chosen for defining a contact plane and the remaining contacts are projected on the plane. The AGP quality measure for such grasp is the area of the CH of the projected contacts on the plane given by

$$Q_{AGP} = Area(CH(\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}, \mathbf{p'_4}, ..., \mathbf{p'_n})) \tag{12}$$

where $\mathbf{p'_i}$ is the projected point of contact $\mathbf{p_i}$ on the plane formed by $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$. This extension has several problems. First, which three contacts should be chosen to form the contact plane? Choosing different contacts will result in different quality values. This problem has no good solution so far and probably should be done based on the geometry of the end-effector. In our implementation we would randomly choose the three contacts. Another problem involves the complexity of the computation. Computation of the AGP for a 3-finger grasp is fast and efficient; however, as we increase the number of fingers, the computational cost for the projections and the convex-hull increases.
2. **Minimum singular value of the grasp matrix (MSV)**. A grasp is said to be singular if at least one singular value of the grasp matrix G goes to zero. In such case, the grasp loses its ability to counter-balance external wrenches in at least one direction [11]. Therefore, the MSV quality measure is calculated to be the smallest singular value of the grasp matrix. Mathematically,

$$Q_{MSV} = \min(\lambda(G)) \tag{13}$$

where $\lambda(G)$ is the singular values vector calculated by the square roots of the eigenvalues of GG^T.
3. **Largest ball in wrench space (LBW)**. In this method, the grasp quality is equivalent to the radius of the largest ball centered at the origin of the GWS and fully contained in the $CH(\mathcal{W})$ [6, 10]. In other words, the grasp quality measure is defined as the distance from the origin of the GWS to the closest facet of the $CH(\mathcal{W})$. Formally, we can say that the quality measure Q is defined as

$$Q_{LBW} = \min_{\mathbf{w} \in \partial CH(\mathcal{W})} \|\mathbf{w}\| \tag{14}$$

where $\partial CH(\mathcal{W})$ is the boundary of $CH(\mathcal{W})$. The quality measure Q_{LBW} is the radius of the ball and denotes the weakest net wrench of the grasp. This means that the largest contact forces would have to be applied to counter balance an external wrench applied along the weakest direction. The weakest direction is defined by the vector from the origin to the point where the Q_{LBW}-sized ball is tangent to the boundary of $CH(\mathcal{W})$.
This method is the most popular and researched grasp quality measure. However, it has high computational cost. Notice that as we increase the number s of unit vectors discretizing the LFC, the size of $\mathcal{W}$ increases and the computational runtime will be higher. Moreover, this measure depends on the choice of the reference system [12] and usually the center of mass of the object is chosen (if possible) as the origin.
4. **Volume of grasp wrench space (VGW)**. An alternative to the LBW criterion was proposed to eliminate its reference frame dependence [14]. In this quality measure the volume of the wrench space is computed, that is,

$$Q_{VGW} = Volume(CH(\mathcal{W})) \ . \tag{15}$$

This measure is indeed invariant of the reference frame but the indication of the weakest direction for counter-balancing external wrenches was removed. Moreover, similar to the LBW, the runtime is significantly affected by the discretization size of the LFC.

4 Standard Deviation based Quality Measure

One can think of grasping a long beam from one edge. Although the grasp may be force closure, large forces would have to be applied to counter-balance external torques such as one caused by gravity. Hence, as discussed in Section 3, many grasp quality criteria are based on the distribution of the contact points on the surface of the grasped object. A good distribution would result in balanced and relatively low loads on the fingers. We present a novel distribution based quality measure termed Standard Deviation based Quality Measure (SDQM), and compare it to other known measures.

4.1 Proposed quality measure

The main concept of defining the SDQM is by evaluating the distribution of the contact points on the grasped object. We use a Standard Deviation (SD) measure to quantify the distribution.

Given a set of n contact points $\mathcal{P} = \{\mathbf{p_1}, ..., \mathbf{p_n}\}$ where $\mathbf{p_k} = (x_k, y_k, z_k)$, the SD vector is defined to be

$$\tau = \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_z \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{n}\sum_{k=1}^{n}(x_k - \bar{x})^2} \\ \sqrt{\frac{1}{n}\sum_{k=1}^{n}(y_k - \bar{y})^2} \\ \sqrt{\frac{1}{n}\sum_{k=1}^{n}(z_k - \bar{z})^2} \end{pmatrix} \tag{16}$$

where $\bar{\mathbf{p}} = (\bar{x}\ \bar{y}\ \bar{z})^T$ is the mean vector of the respected coordinates of $\mathcal{P}$. The SD vector τ contributes a measure of the distribution of the contacts on the object. However, the values of τ strongly depend on the reference frame of the points in $\mathcal{P}$.

One of the desired properties of the new quality measure is to have it invariant of reference frames. However, it could easily be shown that the SD vector of (16) is non-invariant to rotation. Rotation of the reference frame changes the components in τ between some maximal and minimal values. Thus, we are interested in evaluating τ to be the extremum of the SD. A simple and effective way to do so is using Principal Component Analysis (PCA) [8]. The PCA method is used to compute the most meaningful basis of a data set. That is, it will output the basis coordinate frame such that the x-axis has the highest SD and the other two axes in turn will have the highest SD possible under the orthogonality constraint. Therefore, the rotation matrix

$$R_{SD} = PCA(\mathcal{P}) \tag{17}$$

is the output of the PCA used to rotate the vectors in $\mathcal{P}$ to the meaningful basis in terms of the SD. Note that the Singular Value Decomposition (SVD) [32] could also be used. However, the runtime of the PCA is better than that of the SVD.

Given the rotated vector set $\mathcal{P}' = \{\mathbf{p'_1}, ..., \mathbf{p'_n}\}$ such that $\mathbf{p'_k} = R_{SD} \cdot \mathbf{p_k} = (x'_k\ y'_k\ z'_k)^T,\ k = 1, ..., n$, we define the PCA-SD vector to be (similar to (16))

$$\tau' = \begin{pmatrix} \tau'_x \\ \tau'_y \\ \tau'_z \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{n}\sum_{k=1}^{n}(x'_k - \bar{x}')^2} \\ \sqrt{\frac{1}{n}\sum_{k=1}^{n}(y'_k - \bar{y}')^2} \\ \sqrt{\frac{1}{n}\sum_{k=1}^{n}(z'_k - \bar{z}')^2} \end{pmatrix} \tag{18}$$

where $\bar{\mathbf{p}}' = (\bar{x}'\ \bar{y}'\ \bar{z}')^T$ is the mean vector of the respective coordinates of $\mathcal{P}'$. We now want to show that the new PCA-SD vector is invariant to rotation. First, the next lemma shows that two sets of points, one rotated relative to the other, have equal PCA-SD vectors.

Lemma 1 *Given two sets $\mathcal{P}_1 = \{\mathbf{a_1}, ..., \mathbf{a_n}\}$ and $\mathcal{P}_2 = \{\mathbf{b_1}, ..., \mathbf{b_n}\}$, where there exists a rotation matrix $R \in SO(3)$ such that $\mathbf{b_k} = R \cdot \mathbf{a_k}$ for all $k = 1, ..., n$. The rotation matrices $R^{\mathbf{a}}_{SD}$ and $R^{\mathbf{b}}_{SD}$ are computed by the PCA function such that $\mathbf{a'_k} = R^{\mathbf{a}}_{SD}\mathbf{a_k}$ and $\mathbf{b'_k} = R^{\mathbf{b}}_{SD}\mathbf{b_k}$. By that, the respected vectors are equal; that is, $\mathbf{a'_k} = \mathbf{b'_k}$ for all $k = 1, ..., n$.*

The proof of Lemma 1 is relegated to the Appendix. The next Theorem shows that the PCA-SD vector is invariant to reference frame rotation and translation.

Theorem 2 *Given two sets $\mathcal{P}_1 = \{\mathbf{a_1}, ..., \mathbf{a_n}\}$ and $\mathcal{P}_2 = \{\mathbf{b_1}, ..., \mathbf{b_n}\}$, where there exist a rotation matrix $R \in SO(3)$ and a translation vector $\mathbf{d} \in \mathbb{R}^3$ such that $\mathbf{b_k} = R \cdot \mathbf{a_k} + \mathbf{d}$ for all $k = 1, ..., n$. Let $\tau'_{\mathbf{a}}$ and $\tau'_{\mathbf{b}}$ be the PCA-SD vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Therefore, both PCA-SD vectors are invariant to any arbitrary rotation R and translation d such that $\tau'_{\mathbf{a}} = \tau'_{\mathbf{b}}$.*

The proof of Theorem 2 is relegated to the Appendix. Therefore, based on Theorem 2, the PCA-SD is invariant to reference frame representation.

The new PCA-SD vector τ' expresses the distribution of the grasp points relative to their mean. However, a generalization of a single number for the grasp quality measure is required. A good approach is to have a simple mean of the SD vectors components given by

$$Q_{SDQM} = \frac{1}{3}\left(\tau'_x + \tau'_y + \tau'_z\right) \tag{19}$$

where $\tau'_x, \tau'_y, \tau'_z$ are the components of τ'. In this case, the mean SD of the grasped points was acquired giving a measure of the grasps' distribution. It allows comparison of grasps on the same object and between several objects. The value itself is in unit length. However, for a more intuitive measure and for better comparison of grasps of the same object, we normalize the SDQM by the characteristic lengths of the grasped object. That is, the standard deviation based quality measure is given by

$$Q_{SDQM} = \frac{1}{3}\left(\frac{\tau'_x}{b_x} + \frac{\tau'_y}{b_y} + \frac{\tau'_z}{b_z}\right) \tag{20}$$

where b_x, b_y and b_z are the characteristic lengths of the object. The characteristic lengths are basically the edges of the bounding box of the object as seen in Figure 1. They could be acquired by direct measurements or by a designated algorithm. Given a uniform mesh of

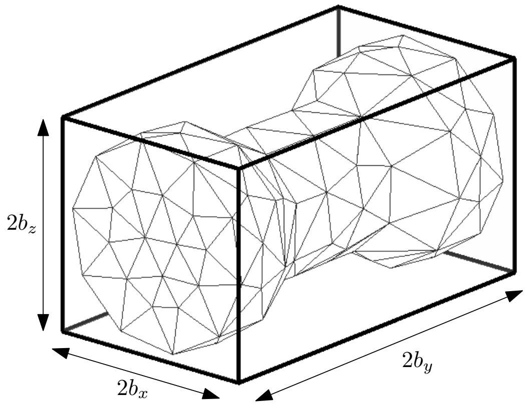


Fig. 1 Bounding box of an arbitrary object.

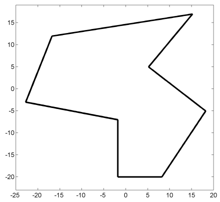
the object, using PCA on the mesh points we can bring the object to its principal axes and find the bounding lengths in each axis, which are the characteristic lengths bounding the object.

The SDQM computed in (20) is now normalized to the object's dimensions such that $Q_{SDQM} \in [0, 1]$. Therefore, the new measure is a comparative tool for grasps of an object where grasps with a value closer to one are better distributed along the object than ones with values closer to zero.

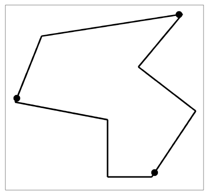
4.2 Example on a 2D shape

In this section we present an example of usage on a simple planar shape including a comparison of the SDQM criterion to other quality measures. An arbitrary polygonal object was selected (Figure 2a) for examination of the proposed measure. The shape was uniformly discretized to 355 points along its boundaries. Next, 3-finger arbitrary grasps were sampled and checked for force closure until acquiring 15,000 force-closure grasps [23]. For each, the SDQM value was calculated as well as the common measures presented in Section 3 for comparison and analysis. The best grasps with the highest quality measure for SDQM, LBW, MSV, VGW, and AGP are presented in Figures 2b-2f, respectively. It should be mentioned that contact points on the corners were not allowed but contact points near the corners were allowed for generality although they might be unfeasible. In Table 1 the quality values for the grasps in Figure 2 are presented. Notice that both SDQM and AGP have graded the same grasp with the highest measure. This is due to the high correlation between the two measures as seen next.

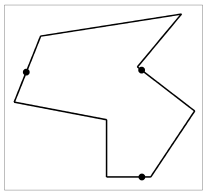
A statistical correlation analysis was conducted to examine the correlation between the discussed criteria. The Pearson product-moment correlation [22] values between criteria computed on the planar object can be



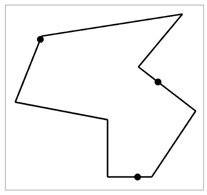
(a) Tested planner object.



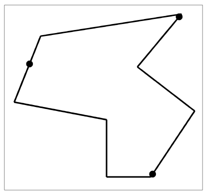
(b) SDQM.



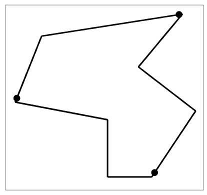
(c) LBW.



(d) MSV.



(e) VGW.



(f) AGP.

Fig. 2 Highest quality grasps of an arbitrary planar object.

Quality of grasp in Fig.	2b	2c	2d	2e	2f
Q_{SDQM}	0.780	0.583	0.594	0.754	0.780
Q_{LBW}	0.233	0.652	0.540	0.342	0.233
Q_{MSV}	1.217	1.550	1.625	1.115	1.217
Q_{VGW}	28.056	24.118	24.289	30.946	28.056
Q_{AGP}	609.3	318.8	311.6	575.6	609.3

Table 1 Quality measures for the grasps presented in Figure 2. The values between different quality measures are not comparable.

seen in Table 2. Moreover, the quality distribution of the SDQM was plotted with the other criteria as seen in Figure 3. Both table and figure show high correlation of the SDQM with the AGP and relatively high correlation with the VGW. These criteria measure the distribution of the contact points on the grasped object and therefore high correlation is expected. This is also the reason that in most computations, both the SDQM and the AGP will output the same grasp as the best one. That is, give the highest quality value to the same grasp. A low correlation of the SDQM with the LBW and the MSV is seen because these criteria measure a different aspect of the grasp such as force directions or singularity.

The runtime aspect of the new measure was also analyzed and compared to other known criteria. In Figure 4 the average runtime for each criterion is presented for

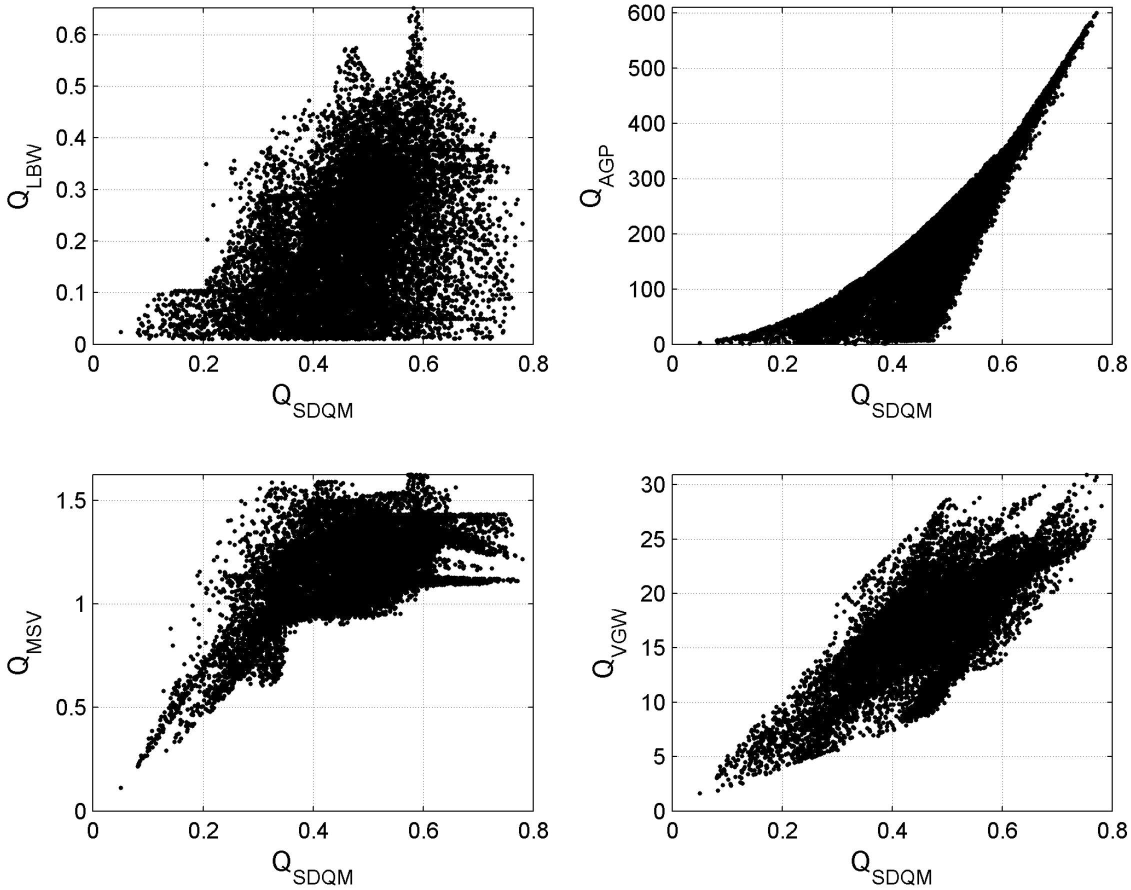


Fig. 3 Correlation of the SDQM to other known quality measures.

	SDQM	AGP	MSV	LBW	VGW
SDQM	1.00	0.86	0.55	0.36	0.78
AGP		1.00	0.38	0.30	0.71
MSV			1.00	0.21	0.53
LBW				1.00	0.40
VGW					1.00

Table 2 Pearson product-moment correlation coefficient of tested quality measures.

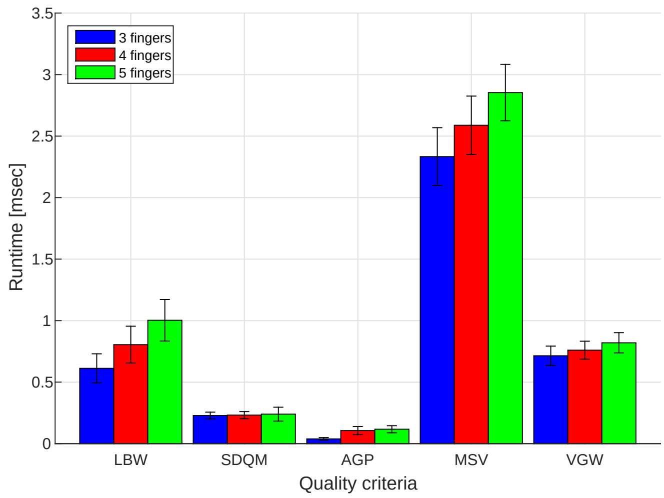


Fig. 4 Average calculation runtime of the grasp quality criteria with their standard deviation.

3-, 4-, and 5-finger grasps. It can be seen that the runtime for the SDQM and the AGP is relatively low compared to the other methods. This is due the low computational complexity of the two methods compared to the others. Moreover, as we increase the number of fingers in a grasp, the runtime increases significantly in all of the measures but the SDQM, which preserves a relatively constant runtime of about 0.2 milliseconds. In this case, the runtime advantage is given to the AGP. However, in the case of spatial objects as will be presented in the next section, we will see a runtime advantage to the SDQM.

5 Grasping of Sheet Metal Parts

The grasp of an SMP is done using sheet metal clamps (illustration in Figure 5a). Clamping of an SMP provides two opposing and collinear forces at each contact point. In this section we present the implications of this property on the grasping model and the application of the SDQM on grasp synthesis of an SMP.

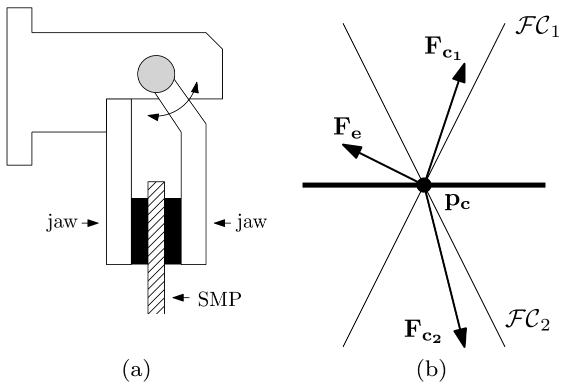


Fig. 5 (a) A sheet metal clamp and (b) an illustration of a single contact point while grasping a sheet metal part using a clamp.

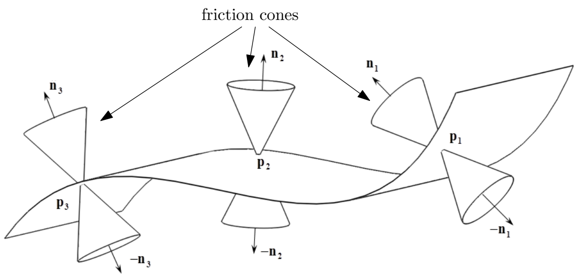


Fig. 6 Grasping of a sheet metal part.

5.1 Force-Closure

As discussed in Section 2.2, a grasp is said to be force-closure if it is possible to apply wrenches at the contacts such that any external forces and torques acting on the part can be counter-balanced. The determination of whether a grasp is force-closure is usually done according to Theorem 1. However, it is a computationally expensive method and in the next theorem we show it is unnecessary while grasping SMPs. First we define a clamping contact.

Definition 1 *Contact point* $\mathbf{p_c}$ *is called a* ***coupled contact*** *if two opposing forces* $\mathbf{F_{c_1}}, \mathbf{F_{c_2}}$ *are applied at point* $\mathbf{p_c}$ *such that*

$$\begin{array}{l} \mathbf{F^N_{c_1}} = \alpha_1 \hat{\mathbf{n}}_\mathbf{c},\ \mathbf{F^N_{c_2}} = -\alpha_2 \hat{\mathbf{n}}_\mathbf{c} \\ \mathbf{F^T_{c_1}} = \beta_1 \hat{\mathbf{t}}_\mathbf{c},\ \mathbf{F^T_{c_2}} = -\beta_2 \hat{\mathbf{t}}_\mathbf{c} \end{array} \tag{21}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$*, where* $\mathbf{F^N_{c_i}}$ *and* $\mathbf{F^T_{c_i}}$ $(i = 1, 2)$ *are the normal and tangential forces, respectively, at contact point* $\mathbf{p_c}$*,* $\hat{\mathbf{n}}_\mathbf{c}$ *and* $\hat{\mathbf{t}}_\mathbf{c}$ *are the normal and tangential unit vectors to the surface at* $\mathbf{p_c}$*.*

The next lemma shows that any external force applied at the contact can be balanced by the coupled contact forces.

Lemma 2 *Given a coupled contact* $\mathbf{p_c}$*, there exists a closure force of the clamp that will counter-balance any external force applied at the point of the coupled contact.*

Proof Let $\mathbf{F_{c_1}}$ and $\mathbf{F_{c_2}}$ (Figure 5b) be the contact forces exerted by the two jaws of the clamp such that

$$\mathbf{F_{c_i}} = \mathbf{F^N_{c_i}} + \mathbf{F^T_{c_i}},\ i = 1, 2. \tag{22}$$

To avoid slippage, the friction cone constraint (1) must be satisfied, that is,

$$\left\|\mathbf{F^T_{c_i}}\right\| \leq \mu \left\|\mathbf{F^N_{c_i}}\right\|,\ i = 1, 2. \tag{23}$$

Assume that an external force $\mathbf{F_e}$ is applied at contact point $\mathbf{p_c}$ such that $\mathbf{F_e} = \mathbf{F^N_e} + \mathbf{F^T_e}$, where $\mathbf{F^N_e}$ and $\mathbf{F^T_e}$ are the normal and tangential forces, respectively, at the contact. To achieve equilibrium, the contact forces must satisfy

$$\begin{array}{l} \mathbf{F^N_e} + \mathbf{F^N_{c_1}} + \mathbf{F^N_{c_2}} = 0 \\ \mathbf{F^T_e} + \mathbf{F^T_{c_1}} + \mathbf{F^T_{c_2}} = 0 \end{array} \tag{24}$$

By assuming $\mathbf{F^N_{c_2}} = -\alpha \mathbf{F^N_{c_1}}$ and $\mathbf{F^T_{c_2}} = \beta \mathbf{F^T_{c_1}}$ for $\alpha, \beta > 0$, and using (23) and (24) on the definition of $\mathbf{F_e}$ we can obtain

$$\|\mathbf{F_e}\| \leq \sqrt{(\alpha-1)^2 + \mu^2(1+\alpha)^2} \left\|\mathbf{F^N_{c_1}}\right\|. \tag{25}$$

Hence, there exists a clamp closure force $\mathbf{F^N_{c_1}}$ that would counter-balance any external force $\mathbf{F_e}$. If there exist contact forces within the friction cones which could counter-balance the external force, these are the forces that would be exerted to maintain static equilibrium [19,20]. □

The importance of Lemma 2 is by understanding that the forces forming a coupled contact can counter-balance any external force applied at the point of the coupled contact, meaning that they span the force space $\mathbb{R}^3$. Using this notion, the following theorem is based on the SMP grasp method where in clamping, each contact point is a coupled contact.

Theorem 3 *For* $n \geq 3$ *frictional coupled contact points* $\mathcal{P} = \{\mathbf{p_1}, ..., \mathbf{p_n} |\ \mathbf{p_i} \neq \mathbf{p_j}\ \forall i \neq j,\ i, j = 1, ..., n\}$ *on the surface of the SMP, if there are at least 3 non-collinear contact points* $\mathbf{p_i}, \mathbf{p_j}, \mathbf{p_k} \in \mathcal{P}$*, the frictional forces at* $\mathcal{P}$ *positively span* $\mathbb{R}^6$*, and the grasp is force-closure.*

Proof According to Lemma 2, one coupled contact $\mathbf{p_i}$ spans the force space $\mathbb{R}^3$ and can counter balance any external force. With an additional coupled contact $\mathbf{p_j}$, external torques can be counter-balanced in any direction except about the axis formed by $\overline{\mathbf{p_i p_j}}$; that is, the four frictional forces span only $\mathbb{R}^5$. Therefore, adding an additional coupled contact at point $\mathbf{p_k} = \{\mathbf{p_k} | \mathbf{p_k} \neq \mathbf{p_i} + \gamma(\mathbf{p_i} - \mathbf{p_j}), \forall \gamma\}$ enables applying torque about the $\overline{\mathbf{p_i p_j}}$ axis. Hence, the three contact points $\mathbf{p_i}, \mathbf{p_j}, \mathbf{p_k}$ with six frictional forces positively span $\mathbb{R}^6$. The three coupled contact points $\mathbf{p_i}, \mathbf{p_j}, \mathbf{p_k}$ positively span the wrench space and therefore additional contact points would not affect the proved force-closure property. □

Theorem 3 provides the notion that a grasp of an SMP with $n \geq 3$ clamps (Figure 6), where at least three clamps are non-collinear, is *always force-closure*. Therefore, there is no need for force closure analysis using the convex-hull method. Moreover, there is no need for modeling and linearizing the friction cones, which consume large computation resources. This notion reduces the runtime of grasp search algorithms drastically. However, there is a need for a criterion to be used to filter out undesired grasps. Therefore, we propose the SDQM criterion for quantifying feasible grasps of the SMP. Maximizing the distribution of the contact points on the SMP is crucial for a feasible grasp and therefore the SDQM is a suitable criterion for this application. In the next subsection we present results for finding the best grasp of an arbitrary SMP under different grasp quality criteria.

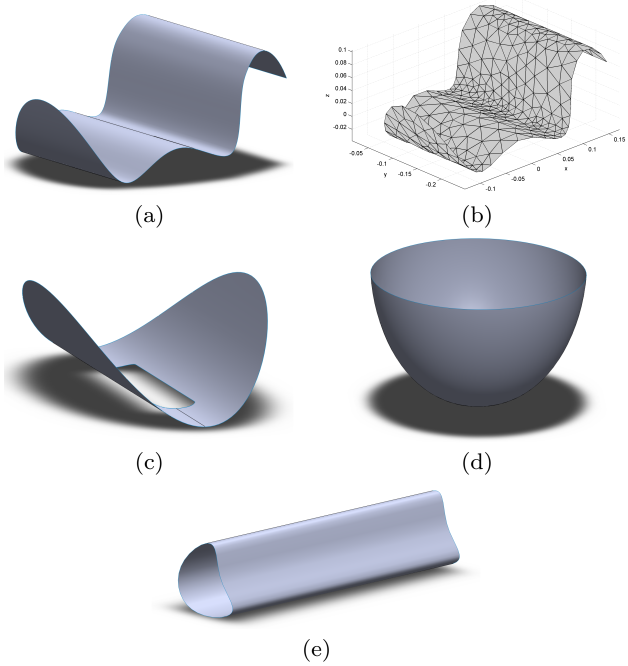


Fig. 7 SMP's used for simulations: (a) simple sheet SMP, (b) its mesh, (c) Folded elliptical SMP, (d) a bowl SMP and (e) a long tube SMP.

5.2 Results and Analysis

For analysis of the SDQM criterion on SMP grasping, we chose an arbitrary SMP CAD model as seen in Figure 7a. Using Comsol Multiphysics the CAD is discretized to 509 mesh triangles as seen in Figure 7b. Due to the fact that an object with such mesh size has about 21.8×10^6 potential 3-clamp grasps, for the analysis we randomly sampled only $15,000$ of them and calculated the quality measures for each to select the best ones according to various criteria.

Figure 8 presents the best grasps according to the different criteria. In this case, the best grasps according to the LBW and VGW criteria are similar and seen in Figure 8a. Similar to the 2D case, the best grasps according to the SDQM and AGP criteria are the same grasp (Figure 8b). This is according to the high correlation between the two criteria as will be discussed next. The best grasp according to the MSV criterion is seen in Figure 8c. The quality values of these grasps can be seen in Table 3. The mentioned similarity between the criteria can also be seen.

The average runtime for calculating each criterion relative to the number of clamps is seen in Figure 9. It can be seen that the LBW, VGW, and MSV criteria have high computation time, which continues to increase as we increase the number of clamps in the grasp. In contrast, the runtimes for the AGP and SDQM are rather low. Moreover, the SDQM runtime is relatively constant at about 121 micro-seconds. A clear runtime advantage of the AGP criterion over the SDQM can be seen in the case of 3-clamp grasps. However, for 4-clamp grasps the advantage is to the SDQM as the AGP runtime ascends over the SDQM and keeps increasing at a moderate slope. The advantage of the SDQM over the other criteria with respect to the number of clamps is in the way it is being computed. Computing the SDQM is done as mentioned by the PCA method, which has two main steps; one is computing the covariance matrix of the contact points and the second is computing the eigenvectors and eigenvalues of the covariance matrix. The first step has a minor runtime effect when increasing the number of contacts. The second step is invariant to the number of contacts as it will always calculate the eigenvectors and eigenvalues of the 3×3 covariance matrix (and a 2×2 matrix in the planar case). In the other quality criteria, the number of contacts in the grasp is much more significant, mostly in those that involve the computation of a convex-hull.

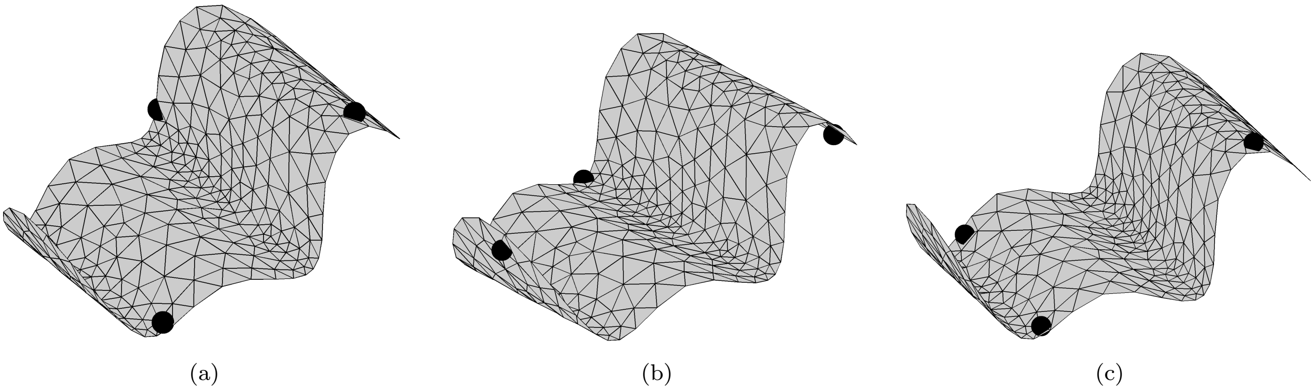


Fig. 8 Best grasp according to (a) the LBW and VGW criteria, (b) the SDQM and AGP criteria and (c) the MSV criterion.

Quality of grasp in Fig.	8a	8b	8c
Q_{SDQM}	0.498	0.521	0.465
Q_{LBW}	0.065	0.059	0.057
Q_{MSV}	0.225	0.218	0.257
Q_{VGW}	0.0018	0.0015	0.0017
Q_{AGP}	0.023	0.025	0.019

Table 3 Quality measures for the grasps presented in Figure 8. The values between different quality measures are not comparable.

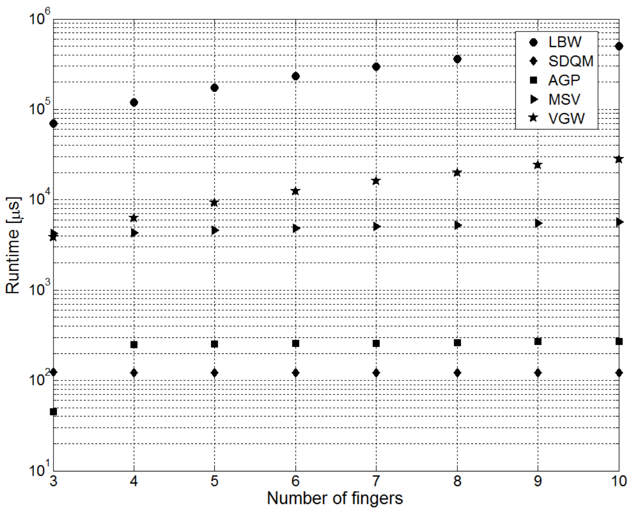


Fig. 9 Average calculation runtime of the grasp quality criteria for SMP grasping.

The quality distribution of the SDQM in the 3-clamp case was plotted with respect to the other criteria distribution as seen in Figure 10. The Pearson product-moment correlation values of the 3-clamp grasp computation shown above are presented in Table 4. The same calculations were done on the ellipse SMP shown in Figure 7c and the tube SMP in Figure 7e. Their correlation results are shown in Tables 5 and 6, respectively. These results show high correlation of the SDQM with the AGP and relatively high correlation with the VGW.

As mentioned in the planar case, these criteria measure the distribution of the contact points on the grasped object and therefore high correlation is expected. This is also the reason that in most cases, both the SDQM and the AGP will output the same grasp as the best one. A relatively high correlation of the SDQM with the LBW and the MSV is seen, contrary to the planar case.

	SDQM	AGP	MSV	LBW	VGW
SDQM	1.00	0.91	0.72	0.74	0.86
AGP		1.00	0.86	0.88	0.95
MSV			1.00	0.94	0.86
LBW				1.00	0.85
VGW					1.00

Table 4 Pearson product-moment correlation coefficient of tested quality measures on the SMP in Figure 7a.

	SDQM	AGP	MSV	LBW	VGW
SDQM	1.00	0.92	0.60	0.67	0.86
AGP		1.00	0.78	0.82	0.95
MSV			1.00	0.95	0.88
LBW				1.00	0.90
VGW					1.00

Table 5 Pearson product-moment correlation coefficient of tested quality measures on the ellipse SMP in Figure 7c.

	SDQM	AGP	MSV	LBW	VGW
SDQM	1.00	0.93	0.57	0.62	0.89
AGP		1.00	0.67	0.73	0.90
MSV			1.00	0.94	0.51
LBW				1.00	0.54
VGW					1.00

Table 6 Pearson product-moment correlation coefficient of tested quality measures on the tube SMP in Figure 7e.

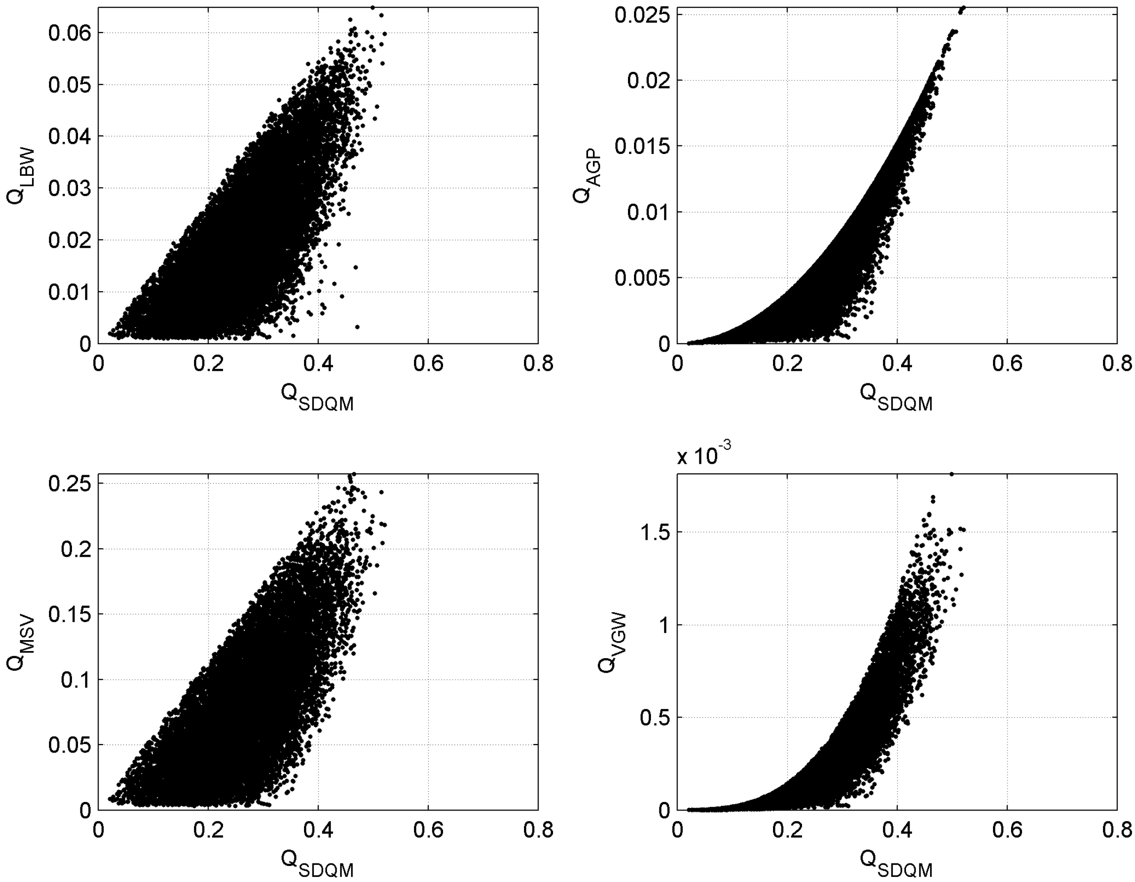


Fig. 10 Correlation of the SDQM on SMPs to other known quality measures based 15,000 grasps sampled on the simple SMP in Figure 7a.

5.3 Stress analysis for grasping SMPs

To examine more benefits of using the SDQM criterion for grasping SMPs, we have analyzed the developed stress on a grasped SMP. The purpose was to show that a distribution based criterion reduces the stresses in the grasped SMP compared to other criteria. In particular, we examined the SDQM criterion.

For that matter, we have generated 550 random SMPs using COMSOL Multiphysics and COMSOL LiveLink with MATLAB. For each SMP we have computed the best four-clamp grasps according to the five quality measures used in the previous section. For each grasp, a 5 g load was applied to the SMP in six directions (x, $-x$, y, $-y$, z, and $-z$), one different direction in each computation. We have acquired the stress distribution at each computation and recorded the maximal von Mises stress on the SMP of the six directions. Thus, we acquired five maximal stresses, one for each criterion, for each of the randomly generated SMPs.

Comparison was conducted to examine which grasp quality criterion yields the minimal stress on the SMP. An example of the results on the SMP in Figure 7c is presented. In Figure 11 the von Mises stress distributions are shown for each grasp. It can be seen that in this case, the SDQM grasp has the minimal maximum stress compared to the other grasps.

Let $\sigma_j(k)$ be the maximal stress on SMP k grasped by a $j = \{SDQM, LBW, AGP, MSV, VGW\}$ criterion computed grasp and let $\sigma_{min}(k)$ be the minimal maximum stress on SMP k such that

$$\sigma_{min}(k) = \min_j \sigma_j(k) \ . \tag{26}$$

Therefore, the relative difference $E_{SDQM}(k)$ of SMP k is given by

$$E_{SDQM}(k) = \frac{\sigma_{SDQM}(k) - \sigma_{min}(k)}{\sigma_{min}(k)} \cdot 100\% \tag{27}$$

and measures the relative distance of the stress computed using an SDQM grasp from the minimal stress

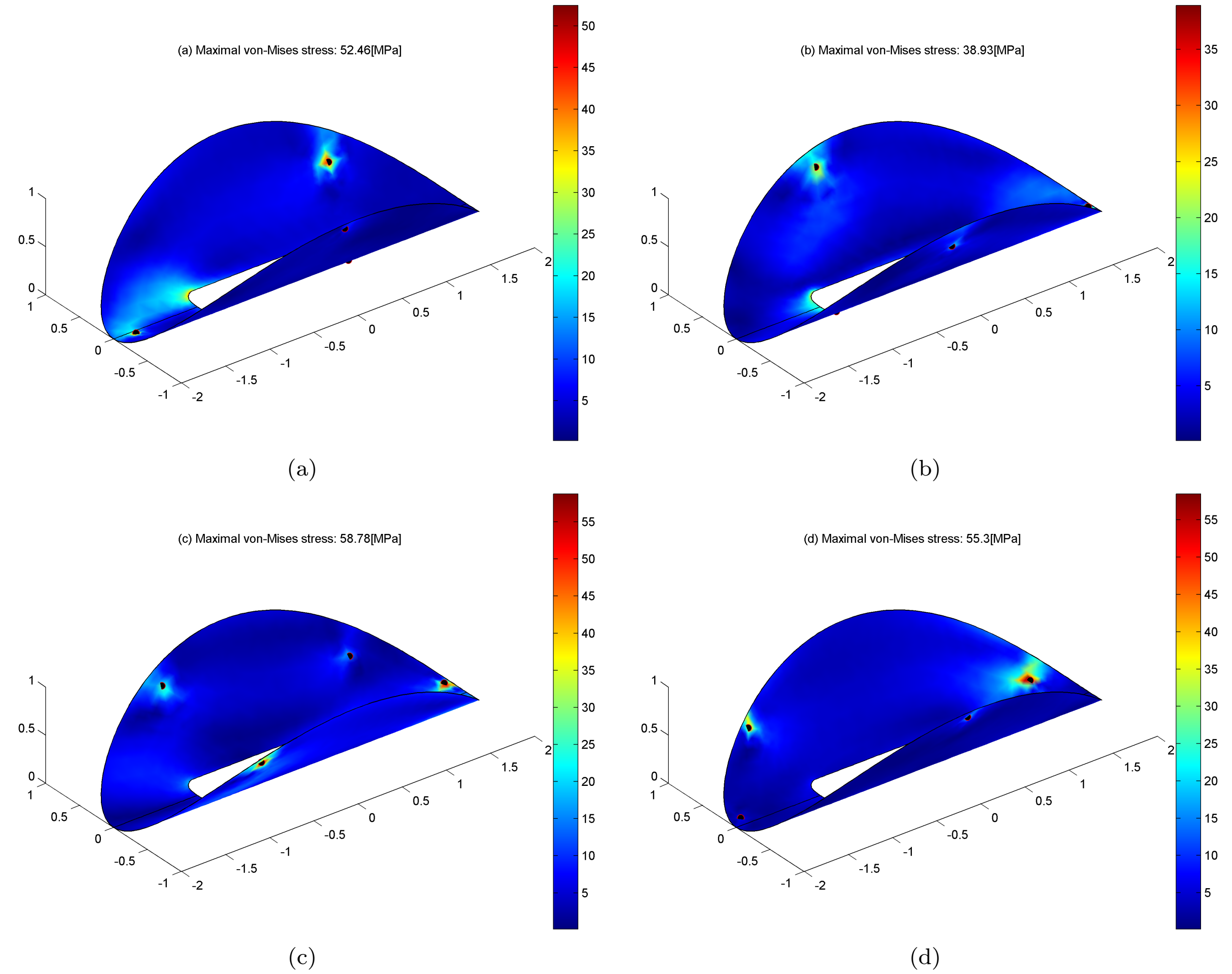


Fig. 11 Example of the stress distribution on an SMP with grasps computed by the (a) LBW and MSV, (b) SDQM, (c) AGP, and (d) VGW quality criteria.

of the stresses computed with the other criteria. The results of E_{SDQM} for all SMPs are presented in the histogram of Figure 12. It can be seen that in 60% of the SMPs, the relative difference of the SDQM stress is below 25%. Meaning, most of the grasps computed with the SDQM criterion yielded grasps with stress close to the lowest stresses computed by the five quality criteria. In fact, in 48% of the SMPs the SDQM criterion computed the grasp with the lowest stress, that is, grasps with $E_{SDQM}(k) = 0$. Figure 13 shows the average relative difference of all criteria. It can be seen that the distribution based criteria are the ones with the lowest average. In particular, the SQDM criterion yielded on average the lowest stresses on the SMPs. The SDQM does not minimize the stresses on the SMP. However, based on these results we can say that on average, the SDQM criterion yields grasps with the lowest stress on the SMP compared to the other presented criteria.

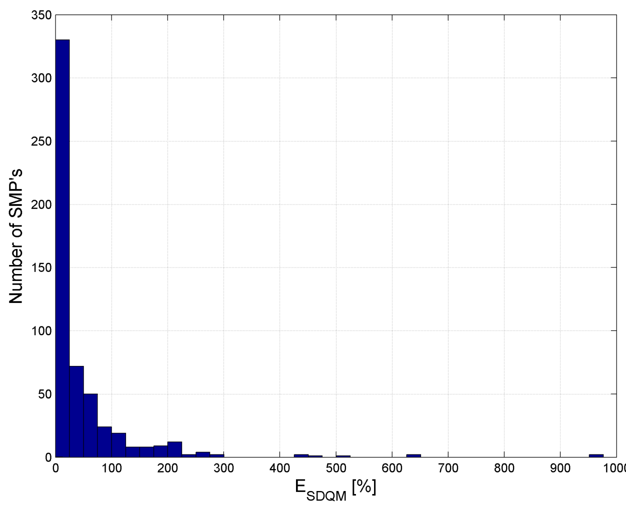


Fig. 12 Relative difference of the SDQM stress to the minimum stress.

SMP	# of clamps	Avg. runtime ±standard deviation [msec]				
		SDQM	AGP	LBW	VGW	MSV
Sheet - Fig. 7a	3	31.73 ±1.66	27.74 ±2.34	24,338 ±144	1,203 ±71.9	1,537 ±94.4
	5	32.57 ±4.3	76.9 ±4.2	65,606 ±3,914	2,865 ±90.9	1,726 ±91.2
	7	34.19 ±3.32	81.68 ±3.32	103,790 ±632	5,027 ±120	1,837 ±78.3
Ellipse - Fig. 7c	3	31.76 ±2.3	20.92 ±2.12	20,513 ±1,240	1,007 ±13.1	1,037 ±89.2
	5	32.06 ±2.46	57.7 ±3.21	51,543 ±631	2,470 ±47.3	1,167 ±25.2
	7	32.5 ±0.44	58.64 ±4.08	87,709 ±6,150	4,430 ±64.6	1,291 ±22.8
Bowl - Fig. 7d	3	32.11 ±3.6	22.98 ±2.8	18,755 ±291	1,748 ±27.6	959 ±48.28
	5	32.24 ±3.5	54.86 ±4.9	53,450 ±1,641	4,852 ±19.26	2,467 ±22.2
	7	32.84 ±3.7	56.41 ±4.1	90,734 ±361	8,827 ±39.2	4,516 ±18.52

Table 7 Average synthesis runtime for several SMPs by generating 300 random grasps.

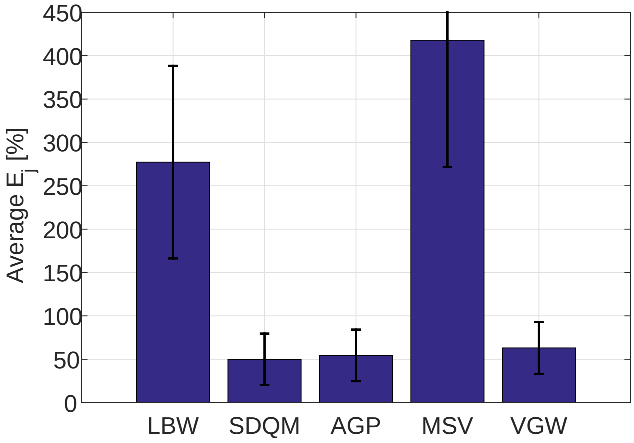


Fig. 13 Average relative difference of the stresses for all computed criteria with their standard deviation.

5.4 Grasp synthesis using SDQM

To validate the use of the SDQM criterion for grasp synthesis we searched for the best grasp for several SMPs. As mentioned, in a mesh of a few hundred points there are myriad grasp candidates. Therefore, we would like to determine the number of grasp candidates to be randomly selected to acquire a high quality grasp. We use the analysis performed in [2]. Note that this analysis can be done with any quality measure, here we use the SDQM measure to demonstrate its synthesis time. For demonstration, we analyze a folded elliptical SMP (Figure 7c) with width of 2 m and longitude of 4 m. We performed 300 grasp search trials and in each trial we generated m random grasps. The grasp with the highest SDQM value was chosen in each trial. Further, in each trial we increased m. Figure 14 presents the results for these simulations for 3-finger and 5-finger grasps. In both cases the highest quality value reaches the average with low variance after only $m = 300$ random grasps. That is, a relatively low number of random points needs to be generated in order to acquire high quality grasps. Experiments have shown that changing the number of mesh triangles describing the object does not significantly changes the results shown in Figure 14. That is, the number of required sampled grasps does not depend on the mesh size.

With this result, we generated a synthesis algorithm where 300 random grasps were generated with a varying number of clamps. Each grasp was evaluated according to different criteria and the highest qualities were recorded. The grasp synthesis was implemented in MATLAB on an Intel-Core i7-2620M 2.5 GHz computer with 8 GB of RAM. This algorithm was implemented on several SMPs and the runtime measurements can be seen in Table 7. It is easy to see that the AGP is runtime superior for a 3-clamp grasp. However, the SDQM is significantly superior as we increase the number of clamps. Both have low runtime by two to three orders of magnitude.

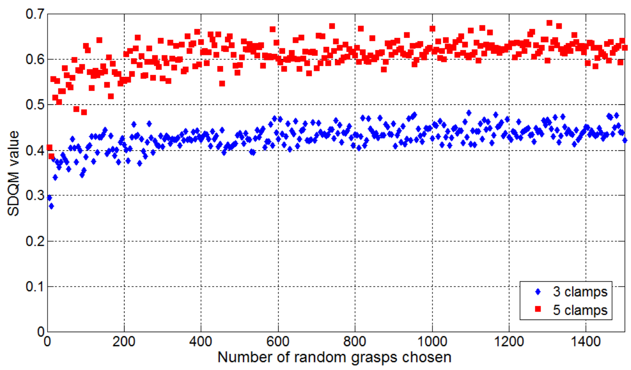


Fig. 14 SDQM value relative to the number of random grasps generated.

It is possible to expand the grasp search by increasing the number m of random grasps generated. However, as seen in Figure 14, increasing m will most likely not improve the quality of the best grasp but only increase the runtime. Although the synthesis runtime is very low, one can reduce the runtime by decreasing m to about 70. That will provide grasps with 77%-84% of the best grasp quality, which is sufficient in most cases. However, such reduction can decrease the runtime by about 78% (approximately 6 milliseconds). The runtime performance could also be improved using parallel computation as the SDQM computation for each candidate grasp is independent.

6 Conclusions

This paper has presented a novel distribution based quality measure termed SDQM. The paper focused on the application of the SDQM for quantifying grasps of SMP. We have shown that the grasping of SMPs is always force closure and therefore a quantitative measure of the grasp is required. Therefore, a comparison of known measures and the proposed one was conducted, focusing on computational runtime and developed stress distribution on the parts.

Simulations have shown high correlation to other known measures that evaluate distribution of the contact points; in particular, the AGP measure has the highest correlation to the SDQM. Performance analysis shows the superiority of the AGP in 3-contacts grasps over other methods in terms of runtime. However, as we increase the number of fingers, the runtime of the AGP increases while the SDQM stays approximately constant. Hence, the application of the SDQM to SMP grasping shows that this approach has the lowest runtime. This is significant in whole arms grasping when a large number of contacts is computed. It is indeed possible to improve the AGP measure in future work and decrease its runtime in more than 3 fingers cases.

It is important to mention that comparisons were made between quality measures of different nature. Some measures are based on the contacts distribution and some take the generated wrenches into account. For SMP, a wrench based measure is not required because the grasp is always force closure and the computation is expensive. In general, the comparison provides a good insight of these different measures in general and in particular to SMP. The correlations, runtime and stress distributions of the different measures can be used by practitioners.

We have shown that grasping of SMP with clamps is always force closure and there is no need for the convex-hull analysis performed in the LBW criterion. However, computing the convex-hull provides useful information such as the magnitude of the maximum resisted wrench in any direction. Hence, future work could use this notion to compute the minimal forces required by the clamps to counter-balance external forces.

An important contribution of the work is the stress analysis which has shown that grasps planned using the SDQM criterion are more likely to yield minimum stresses on the SMP compared to other criteria. Linking between the grasp quality measure and the stress on the grasped part have not been done before. Therefore, this has inspired the authors to consider initiating future research on a minimal stress grasp quality measure.

Appendix

The appendix contains proofs of the results stated in Section 4.

Lemma 1 *Given two sets* $\mathcal{P}_1 = \{\mathbf{a_1}, ..., \mathbf{a_n}\}$ *and* $\mathcal{P}_2 = \{\mathbf{b_1}, \dots, \mathbf{b_n}\}$*, where there exists a rotation matrix* $R \in SO(3)$ *such that* $\mathbf{b_k} = R \cdot \mathbf{a_k}$ *for all* $k = 1, ..., n$*. The rotation matrices* $R^{\mathbf{a}}_{SD}$ *and* $R^{\mathbf{b}}_{SD}$ *are computed by the PCA function such that* $\mathbf{a}'_\mathbf{k} = R^{\mathbf{a}}_{SD}\mathbf{a_k}$ *and* $\mathbf{b}'_\mathbf{k} = R^{\mathbf{b}}_{SD}\mathbf{b_k}$*. By that, the respected vectors are equal; that is,* $\mathbf{a}'_\mathbf{k} = \mathbf{b}'_\mathbf{k}$ *for all* $k = 1, ..., n$*.*

Proof We define the mean vector of $\mathcal{P}_1$ to be

$$\bar{\mathbf{a}} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{a_k} \ . \tag{28}$$

By definition, there exists a rotation matrix R such that $\mathbf{b_k} = R \cdot \mathbf{a_k}$ for all $k = 1, ..., n$. Therefore, the mean vector of $\mathcal{P}_2$ with respect to $\bar{\mathbf{a}}$ is shown to be

$$\bar{\mathbf{b}} = \frac{1}{n}\sum_{k=1}^{n} \mathbf{b_k} = \frac{1}{n}\sum_{k=1}^{n} R\mathbf{a_k} = R\bar{\mathbf{a}} \ . \tag{29}$$

Let $\mathbf{A}$ and $\mathbf{B}$ be matrices concatenating the vectors in $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively, such that

$$\mathbf{A} = \left[\mathbf{a_1} \cdots \mathbf{a_n}\right] , \tag{30}$$

and

$$\mathbf{B} = \left[\mathbf{b_1} \cdots \mathbf{b_n}\right] = \left[R\mathbf{a_1} \cdots R\mathbf{a_n}\right] = R\mathbf{A} \ . \tag{31}$$

The covariance matrices of $\mathcal{P}_1$ and $\mathcal{P}_2$ are given by

$$M_{\mathbf{a}} = \frac{1}{n}\mathbf{A}\mathbf{A}^T - \bar{\mathbf{a}}\bar{\mathbf{a}}^T \tag{32}$$

and

$$M_{\mathbf{b}} = \frac{1}{n}\mathbf{B}\mathbf{B}^T - \bar{\mathbf{b}}\bar{\mathbf{b}}^T \ , \tag{33}$$

respectively. Applying (29) and (31) on (33) yields

$$\begin{aligned} M_{\mathbf{b}} &= \frac{1}{n}R\mathbf{A}\mathbf{A}^T R^T - R\bar{\mathbf{a}}\bar{\mathbf{a}}^T R^T \\ &= R\left(\frac{1}{n}\mathbf{A}\mathbf{A}^T - \bar{\mathbf{a}}\bar{\mathbf{a}}^T\right) R^T \ . \end{aligned} \tag{34}$$

According to (32) we acquire the relation between both covariance matrices to be

$$M_{\mathbf{b}} = RM_{\mathbf{a}}R^T \ . \tag{35}$$

Next, the eigenvalues $\lambda_{\mathbf{a}_i},\ i = 1,2,3$ of $M_{\mathbf{a}}$ are the solution of

$$det\left(M_{\mathbf{a}} - \lambda_{\mathbf{a}} I\right) = 0\ . \tag{36}$$

Similarly, the eigenvalues $\lambda_{\mathbf{b}_i},\ i = 1,2,3$ of $M_{\mathbf{b}}$ are the solution of

$$det\left(M_{\mathbf{b}} - \lambda_{\mathbf{b}} I\right) = 0\ . \tag{37}$$

Equivalently, from (35), the left hand side of (37) can be written as

$$\begin{aligned} det\left(RM_{\mathbf{a}}R^T - \lambda_{\mathbf{b}} I\right) &= det\left(RM_{\mathbf{a}}R^T - \lambda_{\mathbf{b}} RR^T\right) \\ &= det(R) det\left(M_{\mathbf{a}} - \lambda_{\mathbf{b}} I\right) det(R^T)\ . \end{aligned} \tag{38}$$

By definition, matrix R is an orthogonal matrix such that $det(R) = det(R^T) = 1$ and therefore

$$det\left(M_{\mathbf{b}} - \lambda_{\mathbf{b}} I\right) = det\left(M_{\mathbf{a}} - \lambda_{\mathbf{b}} I\right) = 0\ . \tag{39}$$

From (36) and (39) we can conclude that the eigenvalues of $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ are equal. That is,

$$\lambda_{\mathbf{a}_i} = \lambda_{\mathbf{b}_i} = \lambda_i, \quad \forall i = 1,2,3\ . \tag{40}$$

To find the rotation matrices $R^{\mathbf{a}}_{SD}$ and $R^{\mathbf{b}}_{SD}$ we need to compute the eigenvectors of $M_{\mathbf{a}}$ and $M_{\mathbf{b}}$ by solving the following equations

$$(M_{\mathbf{a}} - \lambda_i I)\mathbf{v}_{\mathbf{a_i}} = 0 \tag{41}$$

and

$$(M_{\mathbf{b}} - \lambda_i I)\mathbf{v}_{\mathbf{b_i}} = 0 \tag{42}$$

for $i = 1,2,3$ where $\mathbf{v}_{\mathbf{a_i}}$ and $\mathbf{v}_{\mathbf{b_i}}$ are the respective eigenvectors. Using (35) we can rewrite (42)

$$\begin{aligned} (RM_{\mathbf{a}}R^T - \lambda_i I)\mathbf{v}_{\mathbf{b_i}} &= (RM_{\mathbf{a}}R^T - \lambda_i RR^T)\mathbf{v}_{\mathbf{b_i}} \\ &= R((M_{\mathbf{a}} - \lambda_i I)R^T \mathbf{v}_{\mathbf{b_i}} \\ &= ((M_{\mathbf{a}} - \lambda_i I)R^T \mathbf{v}_{\mathbf{b_i}} = 0 \end{aligned} \tag{43}$$

and from (41) we show that

$$\mathbf{v}_{\mathbf{a_i}} = R^T \mathbf{v}_{\mathbf{b_i}}, \forall i = 1,2,3\ . \tag{44}$$

From its definition, matrix $R^{\mathbf{a}}_{SD}$ is formed by the eigenvectors as follows

$${R^{\mathbf{a}}_{SD}}^T = \left[\mathbf{v}_{\mathbf{a_1}}\ \mathbf{v}_{\mathbf{a_2}}\ \mathbf{v}_{\mathbf{a_3}}\right] \tag{45}$$

and from (44)

$${R^{\mathbf{a}}_{SD}}^T = R^T \left[\mathbf{v}_{\mathbf{b_1}}\ \mathbf{v}_{\mathbf{b_2}}\ \mathbf{v}_{\mathbf{b_3}}\right] = R^T {R^{\mathbf{b}}_{SD}}^T \tag{46}$$

or

$$R^{\mathbf{a}}_{SD} = R^{\mathbf{b}}_{SD} R\ . \tag{47}$$

Applying (47) to $\mathbf{a}'_{\mathbf{k}}$ yields

$$\mathbf{a}'_{\mathbf{k}} = R^{\mathbf{a}}_{SD}\mathbf{a}_{\mathbf{k}} = R^{\mathbf{b}}_{SD} R \mathbf{a}_{\mathbf{k}} = R^{\mathbf{b}}_{SD}\mathbf{b}_{\mathbf{k}} = \mathbf{b}'_{\mathbf{k}}\ . \tag{48}$$

That is, we have shown that

$$\mathbf{a}'_{\mathbf{k}} = \mathbf{b}'_{\mathbf{k}},\ \forall k = 1,...,n\ . \tag{49}$$

We have shown that two sets represented in two rotated reference frames are equal after a PCA rotation. □

Theorem 2 *Given two sets $\mathcal{P}_1 = \{\mathbf{a_1},...,\mathbf{a_n}\}$ and $\mathcal{P}_2 = \{\mathbf{b_1},...,\mathbf{b_n}\}$, where there exist a rotation matrix $R \in SO(3)$ and a translation vector $\mathbf{d} \in \mathbb{R}^3$ such that $\mathbf{b_k} = R \cdot \mathbf{a_k} + \mathbf{d}$ for all $k = 1,...,n$. Let $\tau'_{\mathbf{a}}$ and $\tau'_{\mathbf{b}}$ be the PCA-SD vectors of $\mathcal{P}_1$ and $\mathcal{P}_2$, respectively. Therefore, both PCA-SD vectors are invariant to any arbitrary rotation R and translation d such that $\tau'_{\mathbf{a}} = \tau'_{\mathbf{b}}$.*

Proof The mean vector of $\mathcal{P}_1$ is given in (28) and therefore the mean vector of $\mathcal{P}_2$ can be written as

$$\begin{aligned} \bar{\mathbf{b}} &= \frac{1}{n}\sum_{k=1}^{n}\mathbf{b}_{\mathbf{k}} = \frac{1}{n}\sum_{k=1}^{n}(R\mathbf{a}_{\mathbf{k}} + \mathbf{d}) \\ &= \frac{1}{n}\sum_{k=1}^{n}R\mathbf{a}_{\mathbf{k}} + \frac{1}{n}\sum_{k=1}^{n}\mathbf{d} = R\bar{\mathbf{a}} + \mathbf{d}\ . \end{aligned} \tag{50}$$

The SD vector for $\mathcal{P}_2$ is given by

$$\tau_{\mathbf{b}} = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(\mathbf{b}_{\mathbf{k}} - \bar{\mathbf{b}})^2}\ . \tag{51}$$

According to the definition of $\mathbf{b}_{\mathbf{k}} = R \cdot \mathbf{a}_{\mathbf{k}} + \mathbf{d}$ and using (50), $\tau_{\mathbf{b}}$ with respect to $\tau_{\mathbf{a}}$ has the form

$$\tau_{\mathbf{b}} = \sqrt{\frac{1}{n}\sum_{k=1}^{n}(R\mathbf{a}_{\mathbf{k}} + \mathbf{d} - (R\bar{\mathbf{a}} + \mathbf{d}))^2} = R\tau_{\mathbf{a}}\ . \tag{52}$$

Recall $\mathbf{a}'_{\mathbf{k}}$ and $\mathbf{b}'_{\mathbf{k}}$ to be the PCA rotated vectors of $\mathbf{a}_{\mathbf{k}}$ and $\mathbf{b}_{\mathbf{k}}$, respectively, such that $\mathbf{a}'_{\mathbf{k}} = R^{\mathbf{a}}_{SD}\mathbf{a}_{\mathbf{k}}$ and $\mathbf{b}'_{\mathbf{k}} = R^{\mathbf{b}}_{SD}\mathbf{b}_{\mathbf{k}}$. Therefore, the PCA-SD vector of $\mathcal{P}_1$ is

$$\begin{aligned} \tau'_{\mathbf{a}} &= \sqrt{\frac{1}{n}\sum_{k=1}^{n}(\mathbf{a}'_{\mathbf{k}} - \bar{\mathbf{a}}')^2} \\ &= \sqrt{\frac{1}{n}\sum_{k=1}^{n}(R^{\mathbf{a}}_{SD}\mathbf{a}_{\mathbf{k}} - R^{\mathbf{a}}_{SD}\bar{\mathbf{a}}')^2} = R^{\mathbf{a}}_{SD}\tau_{\mathbf{a}} \end{aligned} \tag{53}$$

and similarly, the PCA-SD vector for $\mathcal{P}_2$ is $\tau'_{\mathbf{b}} = R^{\mathbf{b}}_{SD}\tau_{\mathbf{b}}$. The PCA-SD vector for $\mathcal{P}_2$ could be rewritten using (52) as

$$\tau'_{\mathbf{b}} = R^{\mathbf{b}}_{SD} R \tau_{\mathbf{a}} \tag{54}$$

or according to (47) of Lemma 1 and (53) as

$$\tau'_{\mathbf{b}} = R^{\mathbf{a}}_{SD}\tau_{\mathbf{a}} = \tau'_{\mathbf{a}}\ . \tag{55}$$

That is, the PCA-SD vector is invariant to any arbitrary rotation R and translation. □

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